

# Plateau transitions in fractional quantum Hall liquids

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Received 5 October 1999 and Received in final form 19 November 1999

**Abstract.** Effects of backward scattering between fractional quantum Hall (FQH) edge modes are studied. Based on the edge-state picture for hierarchical FQH liquids, we discuss the possibility of the transitions between different plateaux of the tunneling conductance  $G$ . We find a selection rule for the sequence which begins with a conductance  $G = m/(mp \pm 1)$  ( $m$ : integer,  $p$ : even integer) in units of  $e^2/h$ . The shot-noise spectrum as well as the scaling behavior of the tunneling current is calculated explicitly.

**PACS.** 72.10.-d Theory of electronic transport; scattering mechanisms – 73.20.Dx Electron states in low-dimensional structures (superlattices, quantum well structures and multilayers) – 73.40.Hm Quantum Hall effect (integer and fractional)

## 1 Introduction

The fractional quantum Hall (FQH) effect is a phenomenon observed in a two-dimensional electron system subjected to a strong perpendicular magnetic field. Due to the interplay between the strong magnetic field and interactions among the electrons as well as weak disorder, the transverse resistivity shows a plateau behavior [1]. For a filling factor  $\nu = 1/(\text{odd integer})$ , the theory predicts fractionally charged quasiparticles with charge  $q = \nu e$  [2]. Recent shot-noise experiments in a two-terminal FQH system with a point-like constriction or a point quantum contact (QPC) between the edges seem to be consistent with this theoretical prediction [3]. The FQH system which have any experimental relevance should be confined in a finite region enclosed by one or more edges. Due to the presence of strong magnetic field, the low-energy physics of this two-dimensional electron liquid reduces essentially to that of the one-dimensional edge mode. In one dimension it is known that the interaction plays a significant role. The electrons are strongly renormalized so that the Fermi liquid theory breaks down to be replaced by the Tomonaga-Luttinger liquid (TLL) [4]. If one considers spinless electrons in 1D, the power-law decaying correlation functions of TLL are completely characterized by one parameter  $g$  which represents the strength of interaction. Therefore the parameter  $g$  for an interacting electron system in 1D is not universal. On the other hand a remarkable feature of FQH edge mode is that the parameter  $g$  which controls this 1D system is universal, since  $g$  is related to the topological nature of the bulk FQH liquid (FQHL). For the

edge mode of principal Laughlin states, the parameter  $g$  is simply given by the bulk filling factor  $\nu$  [5].

The edge-tunneling experiment in FQH liquids has shown that a chiral TLL is realized at the edge of FQHL [6]. Indeed the chiral TLL theory has succeeded in the description of non-linear  $I - V$  characteristics for  $\nu = 1/(\text{odd integer})$  [7], but it is also true that the edge-tunneling experiment cannot be explained by a naive TLL theory for other filling factors [8,9]. In particular, for the Jain's composite fermion hierarchy states at filling factor  $\nu = m/(mp + \chi)$  ( $m$ : integer,  $p$ : even integer,  $\chi = \pm 1$ ) [10], Wen's chiral TLL theory predicts that there should be  $m$  edge modes corresponding to each composite fermion Landau level [11]. Due to the existence of these internal degrees of freedom the predicted exponent  $\alpha$  for the  $I - V$  characteristics does not fit the experiment:  $\alpha \sim 1/\nu$ .

Although the observed exponent  $\alpha \sim 1/\nu$  for the tunneling into FQHL does not support the hierarchical structure of edge mode, there is another experimental observation which encourages us to work on this theory. It is the suppressed shot-noise measurement at bulk filling factor  $\nu = 2/5$ , *i.e.*, at  $m = 2, p = 2, \chi = 1$  in a constricted two-terminal Hall bar geometry [12]. They observed the transitions of two-terminal conductance from a plateau at  $G = 2/5$  to another at  $G = 1/3$  and finally to  $G = 0$  as the constriction is increased. On the plateau at  $G = 1/3$  they observed a fractional charge  $q = e/3$ , which indicates that the filling factor near the quantum point contact (QPC) is  $\nu = 1/3$ . The experiment clearly indicates a deep connection between the  $\nu = 2/5$  daughter state and the  $\nu = 1/3$  parents state, and therefore seems to support the hierarchy theory at  $\nu = 2/5$ .

This paper studies the tunneling through a QPC at the edge of FQHL. This topic has captured a widespread

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attention both experimentally and theoretically. For the reasons stated above we focus on the filling factor  $\nu = m/(mp + \chi)$ . The FQH systems at those filling factors will provide an interesting arena to study the hierarchical nature of those liquids. We discuss the successive transition between the plateaux of conductance. The sequence begins with the conductance  $G = G_m = m/(mp + \chi)$  in units of  $e^2/h$ , which is identical to the bulk filling factor. We discuss the selection rule for the transitions between different  $G$ 's. Even though the eventual correctness of the hierarchical TLL theory description is yet to be tested, on which we will be based, we insist that it is of importance to make various interesting applications of the theory. It will enable us to compare the experiment with the theoretical predictions, and hence will be useful to judge the correctness of hierarchical picture.

## 2 Model

Our model has a two-terminal Hall bar geometry. The bulk FQHL in the  $xy$ -plane is confined electro-statically on the  $y$ -direction into a finite region:  $-w/2 < y < w/2$ . Each end of this strip is connected to a source (the left terminal) or to a drain (the right terminal). We assume that the bulk FQHL is incompressible at a filling factor  $\nu$ . Therefore the low-energy excitations are allowed only in the vicinity of two boundaries, which constitute the edge modes. The upper (lower) edge mode carries a current from the left (right) to the right (left) terminal, and the total current  $I$  is defined as the difference of the two. Since there is no mechanism of relaxation in the TLL itself, the chemical potential is uniform in the respective edge modes, *i.e.*, the upper (lower) edge mode has a chemical potential equal to that of the source (drain). All the scatterings occur inside the terminals [13]. In the absence of point-like constriction, the two-terminal conductance  $G = I/V$  is quantized at  $G = \nu$  in units of  $e^2/h$ , since the back-scattering between the two edge modes which breaks the momentum conservation is allowed nowhere through the edge, where  $V$  is defined as the source-drain voltage.

Now we go back to the bulk FQHL. We focus on a filling factor  $\nu = m/(mp + \chi)$  in the Jain's composite fermion hierarchy series, where  $m$ : integer,  $p$ : even integer and  $\chi = \pm 1$ . According to the bulk hierarchy structure, there should be  $m$  edge modes, *i.e.*, each edge mode corresponds to a composite fermion Landau level in the bulk. Then the low-energy physics of this electron liquid is controlled by the  $m$ -channel edge mode, which obey the following Lagrangian density [11]

$$\mathcal{L}_{\text{TLL}} = \frac{i}{4\pi} K^{\alpha\beta} \frac{\partial\phi_\alpha^+}{\partial\tau} \frac{\partial\phi_\beta^-}{\partial x} + \frac{1}{8\pi} U^{\alpha\beta} \left( \frac{\partial\phi_\alpha^+}{\partial x} \frac{\partial\phi_\beta^+}{\partial x} + \frac{\partial\phi_\alpha^-}{\partial x} \frac{\partial\phi_\beta^-}{\partial x} \right), \quad (1)$$

where  $\phi^\pm = \phi^u \pm \phi^l$  with  $\phi^u$  ( $\phi^l$ ) being the edge mode propagating near the upper (lower) boundary of the system. The matrix  $K$  in equation (1) could be identified as

the so-called  $K$ -matrix in the bulk, which together with the electromagnetic charge vector  $t$  as well as the vortex charge vector  $l$  completely specify the universal properties of bulk FQHL [14]. In terms of TLL,  $K$  is a matrix generalization of the parameter  $g$ , which was simply given by  $g = \nu$  for the one-channel case (the edge mode of a principal Laughlin state). The standard construction for the  $K$ -matrix at a hierarchical filling factor  $\nu = m/(mp + \chi)$  yields

$$K = K(m, p, \chi) = \chi I_m + p C_m \quad (2)$$

in the unitary basis  $t^T = (1, \dots, 1)$ , where  $I_m, C_m$  are  $m \times m$  identity and pseudo-identity matrices. By a linear transformation one can decompose the modes into charge and pseudo-spin bosons. Each row and column of the matrices corresponds to a Landau level for the composite fermions, *i.e.*,  $\alpha, \beta = 1, \dots, m$ . However it would be fair to comment that it is still a controversial question what the correct construction of the  $K$ -matrix is [15]. The matrix  $U$  in equation (1) is a positive definite matrix, which specifies among others the velocities of the edge modes. For  $\chi = 1$  charge and pseudo-spin modes propagate in the same direction (co-propagate), whereas for  $\chi = -1$  they are counter-propagating, *i.e.*,  $\chi$  stands for the chirality of the edge modes. For the latter case ( $\chi = -1$ ), the interaction between the edge modes can make the conductance non-universal. The observed conductance, on the contrary, seems to be universal. A remedy for this puzzle would be to put disorder along the edge [16]. In the presence of such disorder our conclusions will be modified, however, which will not be discussed in the body of the paper.

Now we introduce the back-scattering by pinching the Hall bar, *i.e.*, by breaking the global translational invariance at  $x = 0$ . Let us think of applying a gate voltage locally in the middle of Hall bar. It squeezes the Hall bar and makes a quantum point contact (QPC) between the two edges. The QPC introduces the tunneling of quasiparticle through the pinched region of Hall bar. In the TLL model it corresponds to a backward scattering and hence could be described by a periodic potential barrier for the bosonic fields [17]. Let us remember that we are focusing on the bulk filling factor  $\nu_{\text{bulk}} = m/(mp + \chi)$ . According to the Jain's composite fermion hierarchy, there should be  $m$  filled composite fermion Landau levels in the bulk, and accordingly  $m$  types of elementary quasiparticles. Each correspond to a vortex-charge vector  $l = l_j$  where  $(l_j)^\alpha = \delta_j^\alpha$  ( $j, \alpha = 1, \dots, m$ ) with  $\delta_j^\alpha$  being unity for  $\alpha = j$  and vanishes otherwise [18]. The fractional charge carried by the quasiparticle  $l$  is given in general as

$$q/e = t^T K^{-1} l = \frac{1}{mp + \chi} \sum_{\alpha=1}^m l^\alpha. \quad (3)$$

For the elementary quasiparticles  $l = l_j$  one finds  $q = e/(mp + \chi)$ , which is indeed the smallest possible value. For  $m = 2, p = 2, \chi = 1$ , *i.e.*,  $q = e/5$  they could be identified as the current-carrying particles observed in the recent shot-noise experiment at  $\nu = 2/5$  [12].

The tunneling of quasiparticle on the  $j$ th composite fermion Landau level induces a potential barrier proportional to  $\delta(x) \cos \phi_j^\pm$ . Since the scattering amplitude would be different for different types of quasiparticles, the tunneling of these elementary quasiparticles sums up to the following scattering potential barrier:

$$\mathcal{L}_{\text{tun}}^{(\text{I})} = \sum_{j=1}^m u_j \delta(x) \cos \phi_j^\pm. \quad (4)$$

We call it the scattering potential due to the tunneling of class [I] quasiparticles. In equation (4) we took into account only the ‘‘intra-Landau-level’’ processes.

Now I draw your attention to another class of quasiparticles, which we call class [II]. It consists of  $m$  elementary quasiparticles. The vortex-charge vector assigned to this class [II] quasiparticle is  $l^T = (1, \dots, 1)$ . In this combination of the bosonic fields all the neutral modes cancel and the tunneling of such quasiparticles do not accompany any neutral modes. The fractional charge carried by this quasiparticle is found to be  $q/e = m/(mp + \chi) = \nu$ . In the bosonic language it can be written as  $\phi_c = \phi_1 + \dots + \phi_m$ , which indeed corresponds to the charge mode. The scattering potential due to the class [II] quasiparticle tunneling operator can be written as

$$\mathcal{L}_{\text{tun}}^{(\text{II})} = u \delta(x) \cos \phi_c^\pm. \quad (5)$$

Now our total Lagrangian density reads  $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{TLL}} + \mathcal{L}_{\text{tun}}^{(\text{I})} + \mathcal{L}_{\text{tun}}^{(\text{II})}$ .

In the RG analysis in Section 4, we study the scaling behavior of  $u_j$ 's and  $u$ , which are controlled by the scaling dimensions of the quasiparticle tunneling operators:  $\cos \phi_j^\pm$  and  $\cos \phi_c^\pm$ . They are given by  $\Delta_{\text{I}} = \nu/m^2 + 1 - 1/m$  for the class [I] quasiparticles, whereas  $\Delta_{\text{II}} = \nu$  for the class [II] quasiparticle, where  $\nu = m/(mp + \chi)$  [7]. If the scaling dimension is smaller than 1, the corresponding tunneling amplitude tends to have stronger values as the voltage or the temperature is lowered. It is indeed the case both for  $u_j$ 's and  $u$ . In the parameter region  $\{(m, p) | m \geq 2, p \geq 2\}$  in which we are interested, one can prove that

$$\begin{cases} \Delta_{\text{I}} = \Delta_{\text{II}} & \text{for } \chi = -1, m = 2, p = 2 \quad (\nu = 2/3) \\ \Delta_{\text{I}} > \Delta_{\text{II}} & \text{otherwise} \end{cases}, \quad (6)$$

*i.e.*, the class [II] quasiparticles have a lower scaling dimension in most of the cases, and hence more relevant in the RG sense. Another important observation is that there would be at least two ways how the scattering becomes stronger. One way is, as we have described above, to increase its amplitude. However it would be also possible that higher-order cascade of scatterings becomes important, where the single QPC description is no longer valid. One might have to take into account the resonance in such non-perturbative regime.

### 3 Hypotheses

In Section 2 we gave expressions to the possible tunneling processes through a single QPC in terms of the bosonic field  $\phi_j$  or of its linear combination  $\phi_c$ . We considered two classes of quasiparticles. The class [I] corresponds to the elementary quasiparticles with the smallest fractional charge. The class [II] corresponds to the charge mode:  $\phi_c = \phi_1 + \dots + \phi_m$ . We also compared the scaling dimensions of the two classes of quasiparticle tunneling operators. Although the class [II] quasiparticles have a lower scaling dimension in most of the cases and more relevant in the RG sense, it is not unlikely that the class [II] is negligible for some reason; since they are bound states of  $m$  elementary quasiparticles, they are so scarcely created that the scattering potential (5) could not develop enough to be effective at the energy scales in question despite its relevant scaling dimension. Therefore we are encouraged to consider the following cases:

1. Case [A]: class [II] quasiparticles are negligible for some reason. Furthermore the tunneling amplitudes for each channel  $j$  have different orders of magnitude:

$$u_m \gg u_{m-1} \gg \dots \gg u_1. \quad (7)$$

2. Case [B]: class [II] is still negligible, but some  $u_j$ 's have comparable orders of magnitude;

$$u_{j+1} \gg u_j \sim \dots \sim u_{j-k+1} \gg u_{j-k}, \quad (8)$$

where  $k \geq 2$  is an integer.

3. Case [C]: class [II] is no longer negligible, *i.e.*, the amplitudes  $u$  for the class [II] quasiparticle has a comparable magnitude with those for  $u_j$ 's.

The assumption (7) for case [A] might be justified in a way analogous to the edge-channel argument for integer quantum Hall effect (IQH) [19]. Let us consider the Landau levels for composite fermions. The lowest  $m$  Landau levels are completely filled by the composite fermions, and the chemical potential lies between the  $m$ th and  $(m+1)$ th Landau levels. Towards the edge of the sample each energy level tends to be lifted up by the confining potential. Since the  $j$ th edge mode lies in where the chemical potential crosses the  $j$ th energy level, each edge channels are spatially separated. Therefore the tunneling between the  $m$ th edge modes, the spatially closest ones, is supposed to have a much larger amplitude than the other  $m-1$  channels. It is also the case for  $u_{m-1}$  compared with the remaining  $m-2$  channels and so forth. I would like to mention an experiment which encourages us to employ the assumptions (6). It is the suppressed shot-noise measurement at bulk filling factor  $\nu = 2/5$ , *i.e.*, at  $m = 2, p = 2, \chi = 1$  [12]. They observed the transitions of two-terminal conductance from a plateau at  $G = 2/5$  to another at  $G = 1/3$  and finally to  $G = 0$  as the constriction is increased. On the plateau at  $G = 1/3$  they observed a fractional charge  $q = e/3$ , which indicates that the filling factor near the QPC is  $\nu = 1/3$ . Hence a single-channel edge mode described by a  $1 \times 1$   $K$ -matrix;  $K = 3$  is expected near the QPC [20]. On the other hand

in the region where the physics is completely unaffected by the gate, the matrix  $K$  should be given by  $K = K(2, 2, 1)$ . This experiment not only indicates a deep connection between the  $\nu = 2/5$  daughter state and the  $\nu = 1/3$  parents state. It also implies that  $u_2 \gg u_1$  as well as  $u$  is negligibly small.

We have explained above a physical reason why we are interested in the parameter region (7). However we have another example where the assumption (8) seems to be reasonable. It is the spin-singlet state at  $\nu = 2/3$  ( $m = 2, p = 2, \chi = -1$ ), where  $j = 1, 2$  corresponds to each spin indices instead of each Landau level for the spin-polarized state. If we neglect the Zeeman energy, the theory should be symmetric in terms of the two spin components. Therefore it seems more reasonable to assume in this case as  $u_1 \sim u_2$ , which belongs to our case [B] [21].

## 4 RG flow and critical phenomena

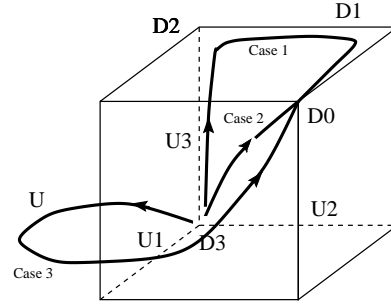
Let us forget for the moment the class (II) quasiparticles. Then we consider the renormalization group (RG) phase diagram in the  $m$ -dimensional space of  $u = (u_1, \dots, u_m)$ . The origin of this plane corresponds to the conductance plateau at  $G = m/(mp + \chi) = \nu_{\text{bulk}}$ . A standard RG analysis shows that only the fixed point at  $u = (\infty, \dots, \infty)$ , is infra-red (IR) stable, since all  $u_j$ 's are found to be relevant. We introduce a small negative gate voltage to the system on the conductance plateau at  $G = G_m$ . We fix the gate voltage so that the scaling at the zero temperature should be controlled by the voltage difference  $qV$  between the two reservoirs. Now we ask where the initial point of our RG flow is.

### 4.1 Successive transitions

Let us consider the case [A]. Our RG flow starts from a point in the vicinity of the origin (unstable fixed point) where equation (7):  $u_m \gg u_{m-1} \gg \dots \gg u_1$  is satisfied. As the voltage is decreased, all  $u_j$ 's scale to larger values. But due to the assumption (7)  $u_m$  increases much faster than the other  $m - 1$  channels. Therefore our RG path flows into the domain  $D_{m-1}$ , where  $D_j$  ( $j = 1, \dots, m - 1$ ) is defined as

$$D_j = \{(u_1, \dots, u_m) | u_m, \dots, u_{j+1} > u_{\text{crtc}} \gg u_j, \dots, u_1\}, \quad (9)$$

$u_{\text{crtc}}$  is a critical value of the tunneling amplitudes such that the phase  $\phi_j$  is pinned when  $u_j > u_{\text{crtc}}$  in order for  $g$  to be quantized. In reality  $u_{\text{crtc}}$  is determined by the strength of impurity potential which could retain the induced quasiparticles at the impurity cite. In the domain  $D_{m-1}$  the effective  $K$ -matrix near the PC reduces to  $K = K(m - 1, p, \chi)$ . Therefore one could identify  $D_{m-1}$  to the plateau of conductance at  $G = G_{m-1}$ . However since the domain  $D_{m-1}$  corresponds to a saddle region of the RG flow, our RG path flows away from  $D_{m-1}$  and goes toward the next saddle region  $D_{m-2}$  defined in the same way as



**Fig. 1.** RG phase diagram in the  $u$ -space for  $m = 3$ ,  $u = (u_1, u_2, u_3)$ . Case 1 represents a RG flow corresponding to the successive transitions which are expected when  $u_3 \gg u_2 \gg u_1$ . Case 2 represents a direct transition to the completely reflecting phase  $D_0$ , which happens when  $u_3 \sim u_2 \sim u_1$ . A direct transition to  $D_0$  is also expected for case 3, where the type (II) scattering potential plays the role.

$D_{m-1}$ . We further introduce the domain  $D_j$  in general for  $j = 1, \dots, m - 1$ . Our RG flow passes through  $D_j$ 's as

$$D_{m-1} \rightarrow D_{m-2} \rightarrow \dots \rightarrow D_1, \quad (10)$$

and finally it flows into the attractive fixed point  $u = (\infty, \dots, \infty)$ , which is identified as the completely reflecting phase  $G = 0$ : the domain  $D_0$  (Fig. 1). The exceptions are the series belonging to  $\chi = -1, p = 2$ , *i.e.*, the  $\nu = 2/3$  state and its daughter states. For those filling factors our RG stops at the  $G = 1$  plateau [20, 22]. In the following we consider the other cases. As the RG path flows from the vicinity of the origin toward the  $G = 0$  phase, the effective  $K$ -matrix near the PC changes as

$$K(m, p, \chi) \rightarrow K(m - 1, p, \chi) \rightarrow \dots \rightarrow K(1, p, \chi) \\ = p + \chi \rightarrow \text{insulator}. \quad (11)$$

Correspondingly we predict the following successive plateau transitions,

$$G_m \rightarrow G_{m-1} \rightarrow \dots \rightarrow G_1 = \frac{1}{p + \chi} \rightarrow 0. \quad (12)$$

One might think the above result is very close to the “global phase diagram” in the quantum Hall effect [23]. Though it indeed is, it differs in that the direction of the transition is specified in our case. Anomalous transitions ( $G_j \rightarrow G_{j-k}$  for  $k \geq 2$ ) are forbidden as far as the assumption (6) is satisfied.

I would like to deduce the scaling behavior of the tunneling current and the shot-noise spectrum on the plateau  $G = G_j$  ( $j = 1, \dots, m$ ). The back-scattering current  $I_b = \nu(e^2/h)V - I$  can be calculated perturbatively with respect to  $u_{j-1}$  and obtained as [5]

$$\langle I_b \rangle = \frac{2\pi q}{\Gamma[2\Delta_I]} |u_{j-1}|^2 \frac{a^{2\Delta_I-2}}{v_c^{2\nu/m^2} v_s^{2(1-1/m)}} (qV)^{2\Delta_I-1}, \quad (13)$$

where  $q = e/(jp + \chi)$  is a fractional charge of the elementary quasiparticle on the plateau  $G = G_j$ , and  $\Delta_I$  is

a scaling dimension of the type (I) quasiparticle tunneling operator:  $\Delta_I = \nu/m^2 + 1 - 1/m$ .  $a$  is a short-distance cutoff, and  $v_c$  and  $v_s$  are velocities of the charge and the pseudo-spin modes, respectively. The shot-noise spectrum

$$S(\omega) = \int_{-\infty}^{\infty} dt \cos \omega t \langle \{I_b(t), I_b(0)\} \rangle \quad (14)$$

is also calculated perturbatively to give,  $S(\omega) = q\langle I_b \rangle (|1 - \omega/qV|^{2\Delta_I-1} + |1 + \omega/qV|^{2\Delta_I-1})$ , which reduces to  $S = 2q\langle I_b \rangle$  in the white-noise limit ( $|\omega| \ll qV$ ) [24]. They are also calculated near the insulating phase to be  $S(\omega) = 2e\langle I \rangle$  with

$$\langle I \rangle = \frac{2\pi e}{\Gamma[2(p+\chi)]} |\tilde{u}_1|^2 a^{2(p+\chi)-2} \tilde{v}_1^{-2(p+\chi)} (eV)^{2(p+\chi)-1}, \quad (15)$$

where  $\tilde{u}_1$  represents the strength of electron tunneling dual to  $u_1$  and  $\tilde{v}_1$  a corresponding velocity.

## 4.2 Anomalous transition

Let us turn to the case [B], where the type [II] quasiparticles are still negligible, but some  $u_j$ 's have comparable orders of magnitude. Here we consider a particular case of the assumption (7); we consider the case where all  $u_j$ 's have comparable orders of magnitude:

$$u_m \sim u_{m-1} \sim \dots \sim u_1. \quad (16)$$

In this case a direct transition from  $G = G_m$  to  $G = 0$  is expected, since all  $\phi_j$ 's tend to be pinned at the same speed. The shot-noise spectrum on the plateau at  $G = G_m$  is given by  $S(\omega) = 2q\langle I_b \rangle$  for  $|\omega| \ll qV$  with

$$\langle I_b \rangle = \frac{2\pi q}{\Gamma[2\Delta_I]} \sum_{j=1}^m |u_j|^2 \frac{a^{2\Delta_I-2}}{v_c^{2\nu/m^2} v_s^{2(1-1/m)}} (qV)^{2\Delta_I-1}. \quad (17)$$

Now we turn our discussion to the insulating phase:  $G = 0$ . Remember each vortex-charge vector  $l$  with integer elements corresponds to a quasiparticle which has a charge given by (3). To construct an electron operator, we have only to set equation (3) to be equal to 1. Of course, there is in principle an infinite number of choice of  $l$  to make it identical to unity. However, as far as the tunneling is concerned, we can pick up most relevant electron operators, which are found to be [18]

$$l = \tilde{l}_j, \quad (\tilde{l}_j)^\alpha = p + \chi \delta_j^\alpha, \quad (18)$$

where  $j = 1, \dots, m$  and each  $\tilde{l}_j$  has  $m$  components, *i.e.*,  $\alpha = 1, \dots, m$ . These electrons look analogous to our class [I] quasiparticles. It indeed is, but we will see that the relation is deeper. Before going into that, the scattering potential barrier due to the tunneling of these ‘‘class [I] electrons’’ can be written as

$$\tilde{\mathcal{L}}_{\text{tun}}^{(I)} = \sum_{j=1}^m \tilde{u}_j \delta(x) \cos \left[ \sum_{\alpha=1}^m (\tilde{l}_j)^\alpha \tilde{\phi}_\alpha^+ \right]. \quad (19)$$

Here the ‘‘inter-Landau-level’’ tunnelings are neglected again. Note that the  $x$ -axis is taken along the edge which is assumed to be completely reflected in the insulating phase.

Let us take notice of the duality between the quasiparticle tunneling and the electron tunneling, which is exact when  $u_m = u_{m-1} = \dots = u_1$  [21]. To see this, let us go back to the weak-scattering phase, *i.e.*, we start with the Lagrangian density:  $\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{TLL}} + \mathcal{L}_{\text{tun}}^{(I)}$ . We starts our RG from the vicinity of the origin in the  $m$ -dimensional space of  $u = (u_1, \dots, u_m)$ . We assume that the condition (16) is satisfied. As the voltage is decreased, all  $u_j$ 's scale to larger values. Then one is encouraged to employ the duality transformation, *i.e.*, one considers the tunneling of instantons between the potential minima [25]. Up to the lowest non-trivial order with respect to those instantons, one obtains a model which has exactly the same form as equation (19), where  $-\tilde{u}_j/2$  corresponds to an instanton fugacity whereas  $\{\tilde{\phi}_\alpha^+\}$  is identified as a set of bosonic fields dual to  $\{\phi_\alpha^+\}$ .

The shot-noise spectrum in the insulating phase is given by  $S(\omega) = 2e\langle I \rangle$  for  $|\omega| \ll qV$  with the tunneling current  $I$  scaling as

$$\langle I \rangle = \frac{2\pi e}{\Gamma[2\tilde{\Delta}_I]} \sum_{j=1}^m |\tilde{u}_j|^2 \frac{a^{2\tilde{\Delta}_I-2}}{\tilde{v}_c^{2/\nu} \tilde{v}_s^{2(1-1/m)}} (eV)^{2\tilde{\Delta}_I-1}. \quad (20)$$

The scaling dimension  $\tilde{\Delta}_I$  of our class [I] electron tunneling operator is given by  $\tilde{\Delta}_I = 1/\nu + 1 - 1/m$ .  $\tilde{v}_c, \tilde{v}_s$  are velocities in the insulating phase.

## 4.3 Direct transition

Let us consider the case [C]. In this case we obtain still different results. In the presence of class [II] quasiparticles, the physics tends to be controlled by the scattering potential  $u$  as the energy in question is lowered. In the region where  $u_1, \dots, u_m \ll u \ll qV$  is satisfied, *i.e.*,  $G \sim G_m$ , one obtains  $S(\omega) = 2q\langle I_b \rangle$  with

$$\langle I_b \rangle = \frac{2\pi q}{\Gamma[2\Delta_{II}]} |u|^2 a^{2\Delta_{II}-2} v_c^{-2\Delta_{II}} (qV)^{2\Delta_{II}-1}. \quad (21)$$

The fractional charge of the class [II] quasiparticle is identical to the bulk filling factor:  $q/e = m/(mp + \chi) = \nu$ , which is also equal to the scaling dimension of the corresponding quasiparticle tunneling operator:  $\Delta_{II} = m/(mp + \chi) = \nu$ . As the RG path flows into the strong-scattering phase, the conductance  $G$  shows a direct transition to  $G = 0$  again. However the scaling behavior of the tunneling current in the insulating phase is less clear. The reason is that the duality is not existing in this case so that the physical interpretation of the strong-scattering phase Lagrangian is lacking.

## 5 Summary: the selection rule

In summary we obtained the following selection rules for the transition between plateaux starting with the bulk

value  $G = m/(mp + \chi) = \nu$ . For case [C] a direct transition to the Hall insulator is expected. For cases [A] (and [B]) successive transitions from one  $G$  to another  $G$  is allowed under the following selection rule:

$$\begin{aligned} g = \nu(j, p, \chi) &\rightarrow g = \nu(j', p', \chi') \\ j' = j - 1, \quad p' = p, \quad \chi' = \chi. \end{aligned} \quad (22)$$

An anomalous transition  $j \rightarrow j - k$  ( $k > 2$ : integer) is expected between the same  $p$  and  $\chi$  when the condition (8) is satisfied.

The overall picture of the system which results from the plateau transitions discussed above is the following. We started with the situation where the filling factor is extended uniformly over the whole system, *i.e.*, equal to the bulk value  $\nu = m/(mp + \chi)$ . Then we effectively increased the gate voltage in units of the voltage difference  $V$  between the two terminals. We found successive transitions of the conductance (12) when the condition (7) is satisfied. The question is what happens between the QPC and the bulk FQHL. Each time  $G$  passes through one plateau ( $G = G_j$ ), there should appear one additional incompressible strip with a filling factor  $\nu = j/(jp + \chi)$ .

Before ending this paper, I mention that the edge-confining potential is assumed to be steep through the paper enough to avoid the complexities which may arise when the confining potential is smooth [26,27]. In conclusion we studied the successive transitions of conductance between different plateaux of hierarchical FQHL. The scaling behavior of the tunneling current and the shot-noise spectrum are calculated perturbatively on each plateau of the conductance. We discussed the selection rules for the transition between different plateaux of the conductance in order that the theory could be tested by the experiments.

This paper is an extension and a generalization of an earlier paper with K. Nomura (Ref. [20]). I am grateful to him for useful discussions and a collaboration. I am also grateful to Y. Morita for a key comment to initiate the present work. I would like to thank P. Lederer and N. Nagaosa for their suggestions and encouragements. I would like to acknowledge the kind hospitality during my participation in the Trimestre Fermions Fortement Correlés (IHP, Paris). I was supported by JSPS Research Fellowships for Young Scientists, and am supported by Ministère de l'Éducation Nationale, de la Recherche et de la Technologie.

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